TAYLOR'S THEORY OF ATMOSPHERIC TURBULENCE.

(Presented at a Physics Department Colloquium, University of Wisconsin.)

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A theory of atmospheric turbulence, verified by numerous quantitative experiments, has been developed in three papers contributed to the Royal Society since 1914, by G. I. Taylor, Schuster reader in meteorology in the University of London in 1914, but since then an officer in the meteorological service of the British Army.

Taylor regards turbulence as made up of eddies and considers an eddy as air that moves from a stratum where it has the same temperature, humidity, and momentum as its surroundings to another stratum, with which it mixes. He makes no effort to separate kinetic and thermal turbulence. He does not, indeed, study individual eddies, although he makes use of such studies by Dines, but his work deals with the effects of turbulence in vertically transferring heat, humidity, and momentum. He neglects the effect of radiation in transferring heat from stratum to stratum, not only in dealing with observations at sea off Newfoundland, where cloudiness perhaps justified the assumption, but also in dealing with average conditions over Paris. His ideas and methods will perhaps be understood from the following summary of his papers:

In his first paper, "Eddy Motion in the Atmosphere," (*Philosophical Transactions*, volume 215, 1915, pages 1-26) Taylor arrives, from consideration of the transference of heat across a large horizontal surface at height z, at the expression

$$\frac{\partial \theta}{\partial t}=\frac{\overline{w}d}{2}\frac{\partial^2\theta}{\partial z^2} \tag{1}$$

for the propagation of heat by means of eddies, in which $\theta(z, t)$ is the average potential temperature of the air in the layer at height z at time t. w is defined by the relation $\frac{1}{2}\overline{w}d$ = average value of $w(z-z_0)$ over a horizontal plane, w being the vertical component of the velocity of the air, d the average height through which an eddy moves from a layer in which it was at the same temperature as its surroundings to the layer with which it mixes. $\frac{1}{2}\overline{w}d$ roughly represents the average vertical velocity of the air in places where it is moving upward. The divisor 2 is inserted because the air at any given point is equally likely to be in any portion of the path of an eddy, so that the average value of $z-z_0$ should be approximately equal to $\frac{1}{2}(d)$.

The similarity of equation (1) to the equation for the propagation of heat in a substance of conductivity κ, specific heat s and density ρ, viz:

$$\frac{\delta T}{\delta t}=\frac{\kappa}{\rho\sigma}\ \frac{\delta^2 T}{\delta z^2}$$

suggests that potential temperature is transmitted upward through the atmosphere by means of eddies in the same way that temperature is transmitted in a substance of conductivity κ. The atmosphere may be assumed, then, to have an eddy conductivity, provided $\frac{\kappa}{\rho\sigma}=\frac{\overline{w}d}{2}$.

The upward propagation of a bend (or inversion) in the temperature-height curve, due to the passage of air across the sharply defined boundary between the warm waters of the Gulf Stream, and the cold Labrador Current, over the Grand Banks of Newfoundland, is the first phenomenon employed by Taylor for the evaluation of the coefficient $\frac{1}{2}\overline{w}d$. The height of the bend was ascertained by Taylor by kite flights from the deck of the ice-scout steamer *Scotia* in the summer of 1913. The time required for the propagation of the bend to its observed altitude was determined by tracing the air back along its trajectory to the boundary between the warm and cold water. The trajectory was obtained by the procedure developed by Shaw and Lempfert in their "Life History of Surface Air Currents" applied to data of wind velocity and direction reported by passing English, German, and Dutch steamers. The temperature of the water was obtained from the weekly charts of sea-surface temperature published by the Meteorological Office. An expression for the rate at which such a bend in the temperature-height curve is propagated upward is obtained by integration of equation (1) for the conditions of the observation. Taylor's solution of this problem gives the relation

$$\tfrac{1}{2}\overline{w}d=\frac{z^2}{4t} \tag{2}$$

where z is the observed height of the bend in the temperature-height curve, t the interval between the sudden change in the rate of change of surface temperature along the air's path, and the time of measurement of the altitude of the inversion.

The following table gives the data and results of seven determinations:

TABLE 1.

Date of observation.	z meters.	t hours.	$\frac{1}{2}\overline{w}d$ C. G. S. units.	Average wind force (Beaufort scale).
May 3, 1913	270	15	3.4×10^3	3.3
July 17, 1913	140	24	$.57\times10^3$	2.0
July 25, 1913	610	168	1.5×10^3	2.0-3.0
July 29, 1913	170	15	1.3×10^3	2.2
Aug. 2, 1913	200	11	2.5×10^3	3.0
Aug. 4, 1913 (two bends in temperature-height curve)	370	36	2.6×10^3	2.5
	770	120	3.4×10^3	3.1

It is to be expected that the turbulence will depend upon the wind velocity, hence it is not surprising to find that on July 17 and 29, when the wind force was about 2, the values of $\frac{1}{2}\overline{w}\,d$ are very much lower than on May 3, and August 2 and 4, when the wind force was about 3. The fact that the figures are so consistent, although t varies from 11 hours to 7 days, and z from 140 meters to 770 meters, indicates that the eddy motion does not diminish to any great extent in the first 770 meters above the surface.

The humidity-height curves obtained from the kite flights from the deck of the *Scotia* show bends at the same heights as the bends in the temperature curves, showing that changes in the amount of water vapor in the atmosphere are propagated upward in the same way as changes in temperature. This is to be expected, for it is evident that the reasoning which was used to deduce equation (1) would serve equally well to deduce an equation $\frac{\partial m}{\partial t}=\frac{\overline{w}d\partial^2 m}{2\partial z^2}$ for the propagation of water vapor into the atmosphere.

In dealing with the upward propagation of momentum by eddy motion it is necessary to take into account the horizontal components of eddy motion, because the eddy can gain or lose velocity owing to the existence of local variations in pressure over a horizontal plane. Such variations are known to exist; they are, in fact, a necessary factor in the production of disturbed motion. Taylor has been unable to solve the problem for motion in three dimensions, but for motions in the x, z, plane, limiting the analysis to incompressible fluids, he finds that, as before, in the case of the eddy conduction of heat, the average value of w' $(z-z_0)$ can be expressed in the form $\frac{1}{2}(\overline{w}d)$ where d is the average height through which an eddy moves before mixing with its surroundings, and $\overline{w}$ roughly represents the average vertical velocity in places where w' is positive. He finds that the effect of the disturbance in reducing the x-momentum is the same as that of a viscosity equal to $\rho\times$ average value of w' $(z-z_0)$, if the motion had not been disturbed. If the same relations hold for three-dimensional motion, then there is a relation $\frac{\kappa}{\rho\sigma}=\frac{\mu}{\rho}=\frac{1}{2}(\overline{w}d)$ between κ the eddy conductivity and μ the eddy viscosity.

Taylor next analyzes the effects of eddy viscosity in preventing the wind from attaining the velocity and direction expected on account of the pressure distribution. The most interesting results of this part of his work are the derivation of an expression for the ratio of the surface wind velocity to the gradient velocity differing essentially from that of Guldberg and Mohn:

$$Q_s/Q_G=\cos\alpha-\sin\alpha \quad \text{(Taylor)}$$
$$Q_s/Q_G=\cos\alpha \quad \text{(Guldberg and Mohn)}$$

and the analytical proof of the fact discovered empirically by Dobson, that the gradient direction is not attained until a height (800 meters) is reached that is more than twice the height (300 meters) at which the gradient velocity is first attained. Taylor's analysis also indicates that above the height at which the gradient direction is attained the wind goes on veering slightly up to a certain height, when it returns again to the gradient direction at a height slightly less than twice the height at which it first attained it. These results of theory are shown to agree closely with the results of observation by Dobson with pilot balloons at the Central Flying School at Upavon, on Salisbury Plain, and with observations by J. S. Dines, while the values of α calculated from Guldberg and Mohn's equation differ by 30° or more from the observed deflection of the surface wind from the gradient wind, and the gradient direction would be attained at the same level as the gradient velocity according to the theory of Guldberg and Mohn. By Taylor's theory the ratio of the height at which the wind direction first becomes the same as the gradient direction H_1 to the height H_2 at which the wind velocity first attains the gradient velocity varies with the deflection of the surface wind from the gradient wind, thus:

α	$\frac{H_1}{H_2}$
0 degrees	3.0
10 degrees	2.8
20 degrees	2.6
30 degrees	2.4
45 degrees	2.2

In Dobson's observations the value of $\frac{H_1}{H_2}$ averaged 2.66, and of α 20°, a remarkable coincidence of theory and observation, a coincidence that indicates that eddy motion does not diminish much in the first 900 meters, in the case of strong winds.

The author derives the following relation between H_1 and the eddy viscosity, for $\alpha=20°$

$$\frac{\mu}{\rho}=\frac{H_1^2\,\omega\sin\lambda}{(2.7)^2} \tag{3}$$

where ω is the angular velocity of the earth 0.000073, and for the regions of observation in the south of England, and the Bank of Newfoundland $\sin\lambda=0.77$. Hence for these regions $\frac{\mu}{\rho}=H_1^2\times0.77\times10^{-5}$.

On land in the case of strong winds $H_1=900$ meters, hence $\frac{\mu}{\rho}=62\times10^3$ in C. G. S. units.

For moderate winds $H_1=800$ meters, and $\frac{\mu}{\rho}=50\times10^3$,

for light winds $H_1=600$ meters, and $\frac{\mu}{\rho}=28\times10^3$.

At sea, assuming that the wind had reached the gradient velocity when it had practically stopped veering with increasing height, H_1 lay between 100 and 300 meters, so that $\frac{\mu}{\rho}$ lay between 0.77×10^3 and 6.9×10^3, values of the same order as those of the values of $\frac{\kappa}{\rho\sigma}$ in Table 1, tending to confirm the theoretical deduction that $\frac{\kappa}{\rho\sigma}=\frac{\mu}{\rho}$.

The author employs the relation $\frac{\mu}{\rho}=\frac{1}{2}\overline{w}\,d$ to determine from observations by J. S. Dines, the size of eddies. In the case taken the average wind velocity was 7 meters per second, the average deviation from the mean vertical velocity 25 cm. per second. d, which is rather less than the average diameter of an eddy, comes out as 40 meters. Since the wind velocity was 7 meters per second, or 420 meters per minute, it is evident that rather less than ten eddies would pass a given spot in a minute. Examination of Mr. Dines' record shows about six peaks per minute on the curve representing vertical velocity, showing that actual observations of eddy motion are in harmony with the assumptions on which the author's theory is based.

The paper closes with a note on the stability of laminar motion of an inviscid fluid. Interest in this question arises from the fundamental disagreement of the conclusion of Reynolds, that the more nearly inviscid the fluid the more unstable it is likely to be, with that of Rayleigh, that instability is impossible when the fluid is quite inviscid. Defining as unstable, motion in which the average value of the square of the distance of any portion of the fluid from the layer out of which the disturbance has removed it, increases with time, the author arrives at the conclusion that the discrepancy between Rayleigh's and Reynolds's work arises from the assumption of perfect slipping at the boundaries in Rayleigh's work, while the complete absence of slipping is assumed in Reynolds's work.

In his second paper, Taylor[1] invokes the aid of the principle of dynamic similarity in testing his theory. In this case it is considered that the tangential force exerted by the wind as it blows over a large tract of land is equal to the skin friction on a similar small surface when subjected to the action of the very high wind which would correspond to the same value of lV/ν (where l represents the linear dimension of the system, V the velocity of the fluid, and ν the kinematic velocity).

[1] *Taylor, G. I.* "Skin friction of the Wind on the Earth's Surface." Proc. Roy. Soc., Ser. A, vol. 92, pp. 196-199.

The trees and houses on the tract of land reduce to the mere roughness on the plate.

For the purpose of comparison the skin friction of the wind is expressed in the form

$$F = \kappa \rho Q_s^2 \qquad (4)$$

Q_s being the wind velocity near the surface, ρ the density of the air, and κ the constant of skin friction.

The highest values of lV/ν obtained by Stanton at the National Physical Laboratory, working with a fluid flowing through a pipe, are expressed by the formula $F = 0.004 \rho V^2$ where V is the velocity of the fluid near the wall, ρ is the density of the fluid, and .004 is the value of the constant κ.

From the equations in his paper on Eddy Motion, Taylor obtains the following expression for κ in the atmosphere

$$\kappa = \frac{2\mu}{\rho} \sin \alpha \frac{\frac{3}{4}\pi + \alpha}{H_1} \frac{Q_g}{Q_s}$$

where the symbols have the same meaning as before. Substituting in this equation the data obtained by Dobson the following values of κ are found:

	$\frac{\mu}{\rho}$	α	H_1	Q_g	Q_s	κ
			Meters.	*cm/sec.*	*cm/sec.*	
Light winds	28×10^3	13°	600	460	330	0.0023
Moderate winds	50×10^3	$21\frac{1}{2}$°	800	910	590	0.0032
Strong winds	62×10^3	20°	900	1,560	950	0.0022

It is concluded (1) that κ does not appear to increase or decrease with wind velocity, a threefold increase in velocity corresponding to a ninefold increase in skin friction. It appears therefore that the skin friction on the earth's surface is proportional to the square of the wind velocity: (2) Since the values of the skin friction coefficient in the atmosphere, 0.002 to 0.003, are of the same order, but slightly smaller than the values found in the laboratory, 0.004, although the scale of the two phenomena differs in the ratio 100,000 to 1, it is evident that the same law of skin friction applies to the friction of the atmosphere on the ground as to small flat plates and pipes. The ratio of the velocity of the fluid near the wall in a pipe to the velocity in the middle, 0.6, is comparable to the ratio of wind velocity near the ground to the gradient wind, which is 0.7 for light winds, 0.6 for strong winds.

A third paper by Taylor deals with the evidence afforded by the daily variation of temperature at various heights on the Eiffel Tower as to the transference of heat by turbulence, and with the relation of turbulence to the daily variation in wind velocity at various heights.

In this paper the power possessed by the atmosphere in virtue of its turbulence of transmitting heat and momentum is represented by the symbol K, which is stated to be roughly equal to the expression $\frac{1}{2}\overline{w}l$ that appeared in the first paper.

The temperature observations on the Eiffel Tower were made during the five years 1890–1894, at altitudes of 123, 197, and 302 meters above ground. Observations at 18 meters are also available from the station on the terrace of the Bureau Météorologique. For the purpose of simplifying the computation the curves of diurnal march of temperature are replaced by the true sine curves which most nearly represent the real curves. Solution of the equation for the convection of heat by turbulence gives the relation

$$K = \frac{\pi}{T} \left(\frac{h}{\log_e R_1 - \log_e R_2} \right)^2$$

where R_1 and R_2 are the ranges at two heights z_1 and z_2, which differ by an amount h, while T is the period of daily variation, 24 hours or 86,400 seconds.

The mean values of K between various heights are shown in the following table:

Month.	1	2	3	4
	18 to 302 meters.	123 to 322 meters.	197 to 302 meters.	18 to 123 meters.
January	4.3×10^4	2.9×10^4	2.7×10^4	11×10^4
February	6.4	4.1	1.6	20
March	10.5	8.3	7.7	24
April	10.2	10.5	8.2	14
May	12.9	14.4	16.7	11
June	18.3	24.4	28.8	12
July	16.7	23.4	30.1	13
August	14.6	13.1	19.6	18
September	8.0	7.2	7.5	10
October	5.9	4.9	5.3	9
November	5.4	3.2	2.5	18
December	6.5	4.4	2.8	15
Mean	10.0×10^4	....	....	....

It will be seen from the table that the turbulence appears to decrease with height in winter, and to increase in summer. This is explained by reference to the vertical temperature gradient of the two seasons. The mean temperature gradient up to 300 meters is considerably less than the adiabatic gradient in winter, and the number of occasions when the adiabatic gradient is comparatively small. Such a gradient, less than the adiabatic, has a tendency to prevent the spontaneous formation of turbulence, and to suppress it when formed by outside agencies, such as obstacles on the ground. In summer the mean temperature gradient in the first 300 meters is much more nearly adiabatic, and the number of occasions when it reaches the adiabatic gradient is large. A gradient equal to the adiabatic has a tendency to encourage the spontaneous formation of turbulence. An increase in the value of K with height in summer is therefore to be expected because K is roughly proportional to the eddy component of turbulent velocity and to the diameters of the eddies.

The values of K near the ground are shown in column 4. These values are less accurate because the method used in deducing K is most liable to error in this case, where the temperature variations are large, and they consequently vary in an apparently haphazard way, yet they show no indication of the annual march so clearly exhibited in all the other columns. The temperature gradient is thought therefore to have little effect on the turbulence in this stratum, which is governed more by the nature of the ground, and by the wind velocity which shows no marked annual march at Paris.

The change in wind direction between the top and bottom of the Eiffel Tower has already been used by Dr. F. Åkerblom to find the viscosity of the atmosphere due to turbulence. This quantity, which is of course equal to K/ρ was found to be about 85 C. G. S. units in winter, and 115 in summer, a difference in the same sense, but considerably less in amount than that indicated by column 1 in the foregoing table. The mean value of K/ρ is given by Dr. Åkerblom as 95 C. G. S. units. Taking the density of the air, ρ, as 0.00125, the value of K from wind velocity measurements comes out

7.6×10^4, which is comparable with the value of 5×10^4 found by Taylor from wind velocity measurements over Salisbury Plain, where the turbulence would be expected from the nature of the ground to be less than over Paris. The agreement between these values and the value of 10×10^4 from the temperatures observed on Eiffel Tower is quite as good as could be expected, considering the approximations in the calculations, and affords satisfactory confirmation of the theory that momentum and heat are transmitted by the same agency, and that the behavior of the lower atmosphere in transmitting heat can be calculated from observations of the retardation of the lower layers of the earth's atmosphere by the friction of the ground.

The remainder of the paper is devoted to the interpretation of the low-level reversal in the type of diurnal march of wind velocity recently brought to light by Hellmann's observations.[2]

The complementary types of diurnal march of wind velocity (1) with a maximum in the middle of the day, observed near the ground, and (2) with a minimum in the middle of the day, on mountains, are well known. Hellmann's observations with anemometers at 2, 16, and 32 meters above the ground show the upper-air type approaching near enough to the ground when the wind is light to give maxima in the middle of the night at 16 and 32 meters. At 16 meters the maxima of midday and midnight are about equal, at 32 meters the night maximum is greater than the day maximum. In strong wind the march is characterized by a midday maximum, and midnight minimum at all of the anemometers.

The Espy-Köppen theory according to which both types of daily variation are the results of midday convectional ascending currents, the circulation of which carry the more stagnant air up from the ground to reduce the velocity of the higher layers, and the faster-moving upper wind down to the ground to increase the velocity of the surface wind, fails to account for Hellmann's observations, because it leads to the conclusion that the vertical currents due to the heating of the ground must extend to a much greater height in strong winds than they do in light winds.

The theory of turbulence, involving frictional as well as convectional interchange of air at different levels affords a satisfactory quantitative explanation of the phenomena observed by Hellmann. In the absence of determinations of the diurnal variation of K, it is necessary to estimate what this would be from the annual variation of K shown by the Eiffel Tower observations. K is assumed to vary continuously from a maximum at midday to a minimum at midnight, in such a way that the corresponding values of the angle α between the directions of the surface wind and the gradient wind vary through the entirely probable range from 10° at midday to 30° at midnight. The velocities are then taken off a graphic representation of the vertical distribution of wind velocity corresponding to a series of values of α. Curves[3] obtained in this way for a series of arbitrary heights agree exceedingly well with the march actually observed by Dr. Hellmann.

The data obtained in various ways as to the value of K are used to estimate the limits to which the ground type of daily march of wind velocity is likely to extend, the results being given in the following table:

[2] Über die Bewegung der Luft in den untersten Schichten der Atmosphäre. Met Zeit, Jan. 1915, vol. 32, pp. 1-16.
[3] Given in the original paper, but not reproduced here.

		Gradient velocity *m. p. s.*	At midday.		At midnight.		Height at which maxima at midday and midnight are equal.
			K	α	K	α	
							Meters.
Strong winds	Summer	13.6	40×10^4	12	6×10^3	28	60
	Winter		13×10^4	17			50
Light winds	Summer	4.6	20×10^4	7	1×10^3	26	30
	Winter		7×10^4	10			25

These theoretical conclusions agree with Dr. Hellmann's observations, in which the reversal in light winds occurred at about 16 meters in winter and about 32 meters in summer. The reversal in strong winds was above all three anemometers, and also above the anemometer 41 meters above the ground at the meteorological observatory at Potsdam.

Taylor's coefficient of eddy conductivity appears to be an important meteorological constant. Its applicability in the dynamics of the lower atmosphere is obvious. That it may be of great practical importance is indicated by the successful elucidation of Hellmann's observations, which are evidently closely related to the phenomenon of the nocturnal inversion of temperature. Observations of it may become essential to the successful forecasting of agricultural frosts.

ATMOSPHERIC STIRRING MEASURED BY PRECIPITATION.

By LEWIS F. RICHARDSON.[1]

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Gentle mixing of a definite portion of air does not alter the total amount of water in it; any increase in the amount must come from water flowing in over the sides of that portion of air. Taking a large horizontal layer of air, and defining *upward flux* as the ratio of amount of water rising across a large horizontal surface in unit time to the area of the surface, we can *define* a coefficient c, such that

$$\varphi = -c \frac{\partial x}{\partial h},$$

where φ is upward flux, h is height, x is amount of water per unit mass of air. When a definite portion of air is removed from one level to another, the total amount of water associated with it does not, of course, change, and hence x does not change, and φ is zero when $\partial x/\partial h$ is zero. It is then very easy to show that

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial p}\left(\xi \frac{\partial x}{\partial p}\right)$$

is the equation for diffusion, where p is pressure, and ξ is equal to $g^2 \rho c$, ρ being density; also that

$$\varphi = -\frac{\xi}{g^2 \rho} \frac{\partial x}{\partial h},$$

ξ being the stirring coefficient, or measure of degree of atmospheric turbulence. Since on the average the water-content of the atmosphere is not increasing, the water which descends as precipitation must have been stirred

[1] Cf. Science Abstracts, Nov. 29, 1919, pp. 502-503.